Fast Algorithm for Symbol Rate Estimation

Suhua TANG†* and Yi YU†, Nonmembers

Summary The cyclic autocorrelation of common digital modulation is researched, and the relationship between the cyclic autocorrelation and the delay, corresponding to the symbol rate, is deduced, then a searching algorithm for the symbol rate is proposed. Theoretical analyses and simulation results show that this method has less computation complexity and is also quite accurate. The estimation result is almost immune to the stationary noise. It’s of practical value to modulation recognition and blind demodulation.

Key words: symbol rate; cyclic autocorrelation; cyclostationarity

1. Introduction

Symbol rate is an important parameter of digital modulation. It’s also one of the essential differences between digital modulation and analogue modulation. Accurate symbol rate estimation is of great importance for modulation recognition, blind demodulation, etc.

Haar wavelet is used to estimate the symbol rate of Phase Shift Key (PSK) modulation and Frequency Shift Key (FSK) modulation in [1]. Differently scaled Haar wavelet transformations are performed on the received signal to detect the discontinuity at the edge of neighboring symbols. Then these results are added together and Fourier transformed, and the position of the maximum is used to estimate the symbol rate. This method works well at the high Signal to Noise Ratio (SNR) situations. However the performance degrades much when SNR is low.

Digital modulation exhibits cyclostationarity because of the symbol period [2], [3]. If the symbol rate of a signal is known, the cyclic autocorrelation and cyclic spectrum can be used to search the signal quickly. The cyclic autocorrelation can also be used to estimate the symbol rate of unknown digital signals. In [4], [5], the characteristic vector consisting of the cyclic autocorrelations corresponding to different delays is calculated, the norm of which is maximized on a certain range to search the symbol rate. The method comes fast into convergence when the dimension of the characteristic vector is big enough. However it costs too much calculation that it’s difficult for application. In this paper, a new method is proposed with less calculation based on the research on the relation between the cyclic autocorrelation of the received signal and the delay.

The rest of this paper is organized as follows. In Sect. 2, the relation between the cyclic autocorrelation and the delay is deduced. Section 3 proposes a fast symbol rate estimation algorithm. Section 4 presents the simulation results. And finally Sect. 5 concludes the paper.

2. Cyclic Autocorrelation of Digital Modulation

Amplitude Shift Key (ASK), PSK, Quadrature Amplitude Modulation (QAM), FSK can all be regarded as a carrier modulated by a complex envelope [6]. When the bandwidth of the modulated signal is unlimited, the envelope of FSK signal is a function of time while the envelopes of ASK, PSK and QAM keep constant during a symbol period. This property can be used to estimate the symbol rate of ASK, PSK and QAM. The relation between the cyclic autocorrelation and the delay of these signals is studied below.

2.1 Complex Envelope of Digitally Modulated Signals

Assume that the modulated signal is

\[ x(t) = a(t)P(t)\cos(2\pi f_c t + \Phi(t)), (n-1)T \leq t < nT \]

where \( a(t) \) and \( \Phi(t) \) are narrow band modulating signals, \( P(t) \) is the narrow band shaping signal (\( P(t) \) stands for the effect of the band-limiting filter), \( T \) is the symbol period and \( f_c \) is the carrier frequency. Hilbert transformation of \( x(t) \) is \( \hat{x}(t) = a(t)P(t)\sin(2\pi f_c t + \Phi(t)) \).

The analytic form of \( x(t) \) lies below

\[ \hat{x}(t) = x(t) + j\hat{x}(t) \]

\[ = [a(t)e^{j\Phi(t)}P(t)]e^{j2\pi f_c t}, (n-1)T \leq t < nT \]  

(2)

Equation 2 is the basic mathematical model of digitally modulated signals. It can represent ASK, PSK, QAM and so on. Here \( A(t) = a(t)e^{j\Phi(t)}P(t) \) is called the complex envelope of \( x(t) \). And \( a_n = a(t)e^{j\Phi(t)}, (n-1)T \leq t < nT \) is kept constant in a symbol period for ASK, PSK, QAM, which correspond to the points in the modulating planisphere. If the points in the modulating planisphere are identically

\[ \begin{align*}
\text{summarize the content of the document to make it more concise and understandable.}
\end{align*} \]
and independently distributed (i.i.d.), complex stationary series \( \{a_n\} \) satisfies the following equation

\[
E\{a_n\} = 0, \ E\{a_n a_m^*\} = \sigma_a^2 \delta_{m,n} \tag{3}
\]

where \( \delta_{m,n} \) is the discrete Dirac delta function. \( \delta_{m,n} = 1 \) when \( m = n \) and \( \delta_{m,n} = 0 \) when \( m \neq n \).

Table 1 gives the common planisphere for 2ASK, 4ASK, 2PSK, 4PSK, 16QAM and 64QAM [6].

### 2.2 Cyclic Autocorrelation of the Received Signal

Assume that the received signal is \( s(t) = x(t) + n_c(t) \), where \( n_c(t) \) is the stationary background noise, \( x(t) = A(t)e^{j2\pi f_c t} \) is the analytic signal in Eq. 2, and \( n_c(t) \) is independent with \( x(t) \). Let \( g(t) \) stand for the shaping signal (in fact it’s the complex impulse response of the transmitting filter, the channel and the receiving filter), then \( A(t) \), the complex envelope of \( x(t) \), can be expressed as follows,

\[
A(t) = \sum_n a_n g(t - nT - \theta) \tag{4}
\]

where \( \{a_n\} \) is the stationary digital series satisfying Eq. 3, \( \theta \) is the delay in a symbol period. The autocorrelation function of \( A(t) \) is

\[
R_A \left( t + \frac{T}{2}, t - \frac{T}{2} \right) = E\{A(t + \frac{T}{2}) A^* (t - \frac{T}{2})\}
\]

\[
= \sum_n \sum_m E\{a_n a_n^*\} g(t + \frac{T}{2} - mT - \theta) g^*(t - \frac{T}{2} - nT - \theta)
\]

\[
= \sigma_a^2 \sum_n g(t + \frac{T}{2} - nT - \theta) g^*(t - \frac{T}{2} - nT - \theta) \tag{5}
\]

From Eq. 5, it can be inferred that \( R_A \left( t + \frac{T}{2} + \frac{T}{2}, t - \frac{T}{2} - \frac{T}{2} \right) = R_A \left( t + \frac{T}{2}, t - \frac{T}{2} \right) \), thus \( A(t) \) is cyclostationary, and its cyclic frequencies set is \( \{\alpha = \frac{k}{T}, k \in Z\} \) (\( Z \) is the integer set); the cyclic autocorrelation corresponding to the cyclic frequency \( \alpha \) is \( (e^{-j2\pi \alpha nT} = 1, \alpha = \frac{k}{T}, k \in Z \) is used during deduction)

\[
R_A^\alpha(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} R_A \left( t + \frac{T}{2}, t - \frac{T}{2} \right) e^{-j2\pi \alpha \tau} dt
\]

\[
= \sigma_a^2 \sum_n \int_{-\infty}^{\infty} g(t + \frac{T}{2} - \theta) g^*(t - \frac{T}{2} - \theta) e^{-j2\pi \alpha (n+\tau)} dt
\]

\[
= \sigma_a^2 e^{-j2\pi \alpha \theta} \int_{-\infty}^{\infty} g(t + \frac{T}{2}) g^*(t - \frac{T}{2}) e^{-j2\pi \alpha \tau} dt \tag{6}
\]

### Table 1 Points of planisphere for ASK,PSK and QAM Modulation.

<table>
<thead>
<tr>
<th>Modulation</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>2ASK</td>
<td>\pm 1</td>
</tr>
<tr>
<td>4ASK</td>
<td>\pm 1, \pm 3</td>
</tr>
<tr>
<td>2PSK</td>
<td>\pm 1</td>
</tr>
<tr>
<td>4PSK</td>
<td>\pm 1, \pm j</td>
</tr>
<tr>
<td>16QAM</td>
<td>\pm 1, \pm j</td>
</tr>
<tr>
<td>64QAM</td>
<td>\pm 1, \pm j</td>
</tr>
</tbody>
</table>

Because \( n_c(t) \) is a stationary process with zero mean, it doesn’t contain nonzero cyclic frequencies. So for the cyclic frequency \( \alpha \neq 0 \), \( s(t) \) and \( x(t) \) have the same cyclic autocorrelation, that’s, \( R_s^\alpha(\tau) = R_s^\alpha(\tau) \).

Now consider the situation where \( \alpha \neq 0 \), the cyclic autocorrelation of the received signal \( s(t) \) is

\[
R_s^\alpha(\tau) = \lim_{\tau \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} R_s(\tau) e^{-j2\pi \alpha \tau} d\tau
\]

\[
eq e^{j2\pi \alpha} \lim_{\tau \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} R_A(\tau) e^{-j2\pi \alpha \tau} d\tau
\]

\[
= e^{j2\pi \alpha} \tau R_s^\alpha(\tau) \tag{7}
\]

So the received signal \( s(t) \) has the same cyclic frequencies as \( A(t) \). And for the cyclic frequency \( \alpha \neq 0 \), \( |R_s^\alpha(\tau)| \) equals \( |R_A^\alpha(\tau)| \).

Now consider a simple case, where the shaping signal \( g(t) \) is the square wave \( U(t) \)

\[
U(t) = \begin{cases} 
1/T, & -T/2 \leq t < T/2, \\
0, & \text{otherwise.}
\end{cases}
\]

Then \( R_A^\alpha(\tau) \) in Eq. 6 can be simplified

\[
R_A^\alpha(\tau) = \sigma_a^2 e^{-j2\pi \alpha \theta} \frac{1}{\pi} \sin(\pi \alpha(T - |\tau|)) / \pi \alpha, |\tau| < T \tag{9}
\]

When \( \alpha = \frac{\pi}{T} \), Eq. 7 can be simplified as follows

\[
|R_s^\alpha(\tau)| = |R_A^\alpha(\tau)| = \frac{1}{\pi} \sigma_a^2 \cos(\pi \alpha(T - |\tau|)), |\tau| < T \tag{10}
\]

From Eq. 10, it can be concluded that for \( \alpha = \pm \frac{\pi}{T} \), \( |R_A^\alpha(\tau)| \) reaches the maximum when \( \tau = \pm T = \pm \frac{2\pi}{\pi} \).

### 2.3 Cyclic Autocorrelation of the Modulated Signal with Limited Bandwidth

In the above, \( g(t) \) is the square wave where the bandwidth of the received signal is unlimited. Next bandwidth limit is considered. Assume that \( g(t) \) is the response of the Linear Time Invariant (LTI) low pass filter \( h(t) \) excited by the square wave \( U(t) \), that is

\[
g(t) = U(t) \otimes h(t) \tag{11}
\]

where \( \otimes \) stands for the operation of linear convolution. It’s obvious that \( g(t) \) given by Eq. 11 is band limited. And \( R_A^\alpha(\tau) \) is the convolution of the cyclic autocorrelation of the series of square waves (Eq. 9), and \( r_h^\alpha(\tau) \), the ambiguity function of \( h(t) \) [2], [3],

\[
R_s^\alpha(\tau) = \frac{1}{\pi} \sigma_a^2 e^{-j2\pi \alpha \theta} \left( \cos(\pi \alpha(T - |\tau|)) \right) \otimes r_h^\alpha(\tau) \tag{12}
\]

where

\[
r_h^\alpha(\tau) = \int_{-\infty}^{\infty} h(t + \frac{T}{2}) h^*(t - \frac{T}{2}) e^{-j2\pi \alpha \tau} dt \tag{13}
\]
If \( h(t) \) is an ideal linear phase low pass filter, and its system function is \( H(f) = e^{-j2\pi f\alpha_0}, -\frac{B}{2} < f < \frac{B}{2} \), Eq. 13 can be simplified
\[
\rho^a_\alpha(\tau) = \int_{-\infty}^{\infty} H(f + \frac{\alpha}{2}) H^*(f - \frac{\alpha}{2}) e^{j2\pi f\tau} df
= e^{-j2\pi \alpha_0} (B - |\alpha|) \sin(\pi \tau (B - |\alpha|)) / \pi \tau (B - |\alpha|), |\alpha| < B \quad (14)
\]

Usually the bandwidth \( B \) of \( h(t) \) satisfies \( \frac{1}{T} < B < \frac{2}{T} \), correspondingly \( s(t) \) has only cyclic frequencies \( \alpha = \pm \frac{1}{T} \), and \( 0 < B - |\alpha| < \frac{1}{T} \). \( \rho^a_\alpha(\tau) \) decreases as \( |\tau| \) increases when \( |\tau| < T \), and it’s a real symmetric function of \( \tau \) (except a complex constant). Then from Eq. 12, it can be concluded that \(|R^a_\alpha(\tau)\)| and \(|R^b_\alpha(\tau)\)| reaches the maximum at \( \tau = \pm \frac{T}{2} \) for \( \alpha = \pm \frac{1}{T} \). Generally only the positive cyclic frequency \( \alpha \) and the positive delay \( \tau \) need consideration for the symmetry.

3. Symbol Rate Estimation

As discussed in Sect. 2, the cyclic autocorrelation \(|R^a_\alpha(\tau)\)| reaches its maximum at \( \tau = \frac{1}{2\alpha} \) when \( \alpha = \frac{1}{T} \). And \(|R^b_\alpha(\tau)\)| approximates 0 when \( \alpha \neq \frac{1}{T} \) and \( \alpha \neq 0 \). Thus the following equation can be used to estimate the symbol rate
\[
\alpha_0 = \frac{1}{T} = \arg \max_{\alpha \in [\alpha_1, \alpha_2]} |R^a_\alpha(\frac{1}{2\alpha})| \quad (15)
\]
where \([\alpha_1, \alpha_2]\) is the searching range of cyclic frequency. Because \( \frac{1}{2\alpha} \) may not be the integer multiple of the sampling period \( T_s \), \( \frac{1}{2\alpha T_s} \cdot T_s \) can be used instead, where \([x]\) is the nearest integer to \( x \). The searching range can be estimated by the spectrum of the nonlinear function, \( s(t + \frac{T}{2}) s^*(t - \frac{T}{2}) \) with a certain delay \( \tau \).

From the above it’s obvious that only the cyclic autocorrelation at \( \tau = \frac{T}{2\alpha} \) is needed for a given cyclic frequency \( \alpha \). While in [4], [5] the characteristic vector of cyclic autocorrelations corresponding to different delays, with the dimension \( 2\gamma + 1 \), is used, where \( \gamma \) is a constant which is generally greater than the responding time of the channel. We replace the vector by a scalar and correspondingly the calculation cost decreases at least \( 2\gamma + 1 \) times.

4. Computer Simulation

The symbol rate estimation simulation is performed for 2ASK, 4ASK, 2PSK, 4PSK, 16QAM, 64QAM signals. The sampling rate is \( F_s = 100KHz \) and the sampling period is \( T_s = 1/F_s \). The symbol rate is \( 400Hz(T = 250T_s) \). The cutting frequency of the low pass filter \( h(t) \) is double the symbol rate. 100 test data, each containing 60 symbols, are generated for each modulation. The additive background noise is stationary and guassian.

Figure 1-6 show the relation between the cyclic autocorrelation \(|R^a_\alpha(\frac{1}{2\alpha})|\) and the cyclic frequency \( \alpha \). From the figures, it is verified that the cyclic autocorrelation \(|R^a_\alpha(\frac{1}{2\alpha})|\) reaches its maximum when the cyclic frequency \( \alpha \) equals the true symbol rate \( \alpha_0 = \frac{1}{T} \).

Figure 7-8 show the standard deviation and maximum error of the estimated symbol rate (normalized by the symbol rate). As SNR decreases, both the standard deviation and the maximum error increase slowly because there are only a finite number of symbols, where the cyclic autocorrelation can’t be fully immune to the stationary background noise. However both of them are still small enough at \( SNR = 0 \).
5. Conclusion

The cyclostationarity of digital modulation was studied thoroughly and the relation between the cyclic autocorrelation and the delay was deduced. Theoretical analysis shows that the cyclic autocorrelation of the received signal equals that of the complex envelope of the modulated signal for the non-zero cyclic frequencies, and this cyclic autocorrelation reaches its maximum when the delay equals half the symbol period. On this basis, a fast algorithm was proposed, and used to estimate the symbol rate of ASK, PSK, QAM signals. Simulation results show that the estimation is very accurate at the low SNR situations. The standard deviation is as low as 0.4Hz at SNR=0 situations when the symbol rate is 400Hz.

The complex envelope of FSK signal is a function of time. However its instantaneous frequency is the pulse amplitude modulated signal. Thus the algorithm proposed above can be used to estimate the symbol rate of FSK signals based on the instantaneous frequency.

References